

CENTRAL ARMENDARIZ RINGS RELATIVE TO A MONOID

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ABSTRACT. In this paper, the notion of central Armendariz rings relative to a monoid is introduced which is a generalization of central Armendariz rings and investigate their properties. It is shown that if R is central reduced, then R is M -central Armendariz for a u.p.-monoid M . For a monoid M and ring R , we prove if R is an M -central Armendariz, then either R is commutative or M is cancellative. Various examples which illustrate and delimit the results of this paper are provided.

1. INTRODUCTION

All rings considered here are associative and unitary. Rege and Chhawchharia [20] introduced the notion of an Armendariz ring. A ring R is called *Armendariz* if whenever polynomials $f(x) = a_0 + a_1x + \dots + a_nx^n, g(x) = b_0 + b_1x + \dots + b_mx^m \in R[x]$ satisfy $f(x)g(x) = 0$, then $a_ib_j = 0$ for each $0 \leq i \leq n, 0 \leq j \leq m$. The name “Armendariz ring” was chosen because Armendariz [4, Lemma 1] had shown that a reduced ring (i.e, a ring without nonzero nilpotent elements) satisfies this condition. Some properties of Armendariz rings and their generalizations have been studied in [20], [4], [3], [7], [24] and [17]. In [19], Liu studied a generalization of Armendariz rings, which is called M -Armendariz rings, where M is a monoid. A ring R is called M -Armendariz (or Armendariz relative to M) if whenever $\alpha = a_1g_1 + \dots + a_ng_n, \beta = b_1h_1 + \dots + b_mh_m \in R[M]$, satisfy $\alpha\beta = 0$, then $a_ib_j = 0$ for each i, j . Some generalizations of Armendariz rings relative to a monoid can be seen in [8], [9], [10], [23] and [25].

According to Agayev, et.al [2], a ring R is called *central Armendariz* if whenever two polynomials $f(x) = a_0 + a_1x + \dots + a_nx^n, g(x) = b_0 + b_1x + \dots + b_mx^m \in R[x]$ satisfy $f(x)g(x) = 0$ then $a_ib_j \in C(R)$ for all i, j . In this paper, we introduce the notion of M -central Armendariz

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rings which are a common generalization of M -Armendariz rings and central Armendariz rings. It is easy to see that the concept of M -central Armendariz rings is related not only to the ring R but also to the monoid M . It is shown that if R is central reduced, then R is M -central Armendariz for a u.p.-monoid M . For a monoid M and ring R , we prove if R is an M -central Armendariz, then either R is commutative or M is cancellative. It is clear that every M -Armendariz ring is M -central. It is shown that the converse is not true in general and the converse is hold if R is a p.p.-ring and M a strictly totally ordered monoid. We end this paper with some applications of M -central Armendariz rings to show there is a strong connection between Baer and p.p.-rings with their monoid rings.

2. M-CENTRAL ARMENDARIZ RINGS

In this section, central M -Armendariz rings are introduced as a generalization of M -Armendariz rings.

Definition 2.1. Let M be a monoid. A ring R is called an M -central Armendariz ring, if whenever elements $\alpha = a_1g_1 + a_2g_2 + \dots + a_mg_m \in R[M]$ and $\beta = b_1h_1 + b_2h_2 + \dots + b_nh_n \in R[M]$ satisfies $\alpha\beta = 0$, then $a_ib_j \in C(R)$ for each $1 \leq i \leq m$ and $1 \leq j \leq n$.

In the following for a monoid M , e always stands for the identity element of M .

Remark 2.2. (1) If $M = \{e\}$, then every ring is M -central Armendariz.

(2) Commutative rings are M -central Armendariz for each monoid M .

(3) If S is a semigroup with multiplication $st = 0$ for each $s, t \in S$ and $M = S^1$, then any noncommutative ring is not M -central Armendariz.

(4) Let $M = (\mathbb{N} \cup \{0\}, +)$. Then a ring R is M -central Armendariz if and only if R is central Armendariz.

(5) Every M -Armendariz ring is M -central Armendariz. But the converse is not true (see Example 2.9).

Recall that a ring R is called central reduced if every nilpotent element of R is central [22]. Let $P(R)$ denote the prime radical and $Nil(R)$ the set of all nilpotent elements of the ring R . The ring R is called 2-primal if $P(R) = Nil(R)$ (See namely [11] and [14]). Also a ring R is called nil-Armendariz relative to a monoid M if whenever elements $\alpha = a_1g_1 + a_2g_2 + \dots + a_mg_m \in R[M]$ and $\beta = b_1h_1 + b_2h_2 + \dots + b_nh_n \in R[M]$ satisfies $\alpha\beta \in Nil(R)[M]$, then $a_ib_j \in Nil(R)$ for each $1 \leq i \leq m$ and $1 \leq j \leq n$ [10].

Recall that a monoid M is called a u.p.-monoid (unique product monoid) if for any two nonempty finite subsets $A, B \subseteq M$ there exists an element $g \in M$ uniquely presented in the form ab where $a \in A$ and $b \in B$ (see [6]).

Theorem 2.3. *Let M be a u.p.-monoid and R a central reduced ring. Then R is a M -central Armendariz ring.*

Proof. If R is a central reduced ring, then R is 2-primal by [22, Theorem 2.15]. Hence $\text{Nil}(R)$ is an ideal of R . Hence by [10, Proposition 2.1], R is nil-Armendariz relative to M . If $\alpha = a_1g_1 + a_2g_2 + \dots + a_mg_m \in R[M]$ and $\beta = b_1h_1 + b_2h_2 + \dots + b_nh_n \in R[M]$ satisfies $\alpha\beta = 0$, then $a_ib_j \in \text{Nil}(R)$ for each i, j . As R is central reduced, $a_ib_j \in C(R)$ for each i, j . \square

Theorem 2.4. *Let M be a monoid and R a ring. If R is an M -central Armendariz, then either R is commutative or M is cancellative.*

Proof. Suppose M is not cancellative. Hence $m, g, h \in M$ are such that $mg = mh$ and $g \neq h$. Then for any $r \in R$ we have $(rm)(1g - 1h) = 0$. As R is M -central Armendariz, $r \in C(R)$. Hence R is commutative. \square

Proposition 2.5. *For a ring R and monoid M with $|M| \geq 2$, the following are equivalent:*

- (1) R is M -central Armendariz;
- (2) R is Abelian, fR and $(1 - f)R$ are M -central Armendariz for any idempotent $f \in R$;
- (3) There is a central idempotent $f \in R$ such that fR and $(1 - f)R$ are M -central Armendariz.

Proof. (1) \Rightarrow (2) Clearly, subrings of any M -central Armendariz ring are M -central Armendariz. It suffices to show R is Abelian. Let $f = f^2 \in R$. Let e be identity of M and $e \neq g \in M$. Then $(fe - fr(1 - f)g)((1 - f)e - fr(1 - f)g) = 0$ implies that $fr(1 - f)$ is central, because R is M -central Armendariz. Hence $fr(1 - f) = 0$. Similarly, $(1 - f)rf = 0$. Therefore f is central.

(2) \Rightarrow (3) is clear.

(3) \Rightarrow (1) Let f be a central idempotent of R and $\alpha = a_1g_1 + a_2g_2 + \dots + a_mg_m$, $\beta = b_1h_1 + b_2h_2 + \dots + b_nh_n \in R[M]$ satisfies $\alpha\beta = 0$. Clearly, we can prove fa_ib_j and $(1 - f)a_i(1 - f)b_j$ are central for each $1 \leq i \leq m$ and $1 \leq j \leq n$. As $R = fR \oplus (1 - f)R$ and f is central, $a_ib_j = fa_ib_j + (1 - f)a_ib_j$ is central for each $1 \leq i \leq m$ and $1 \leq j \leq n$. Thus R is M -central Armendariz. \square

Corollary 2.6. [19, Proposition 3.2] *Every M -Armendariz ring with $|M| \geq 2$ is Abelian.*

The next example shows that the converse of corollary 2.6 is not true in general.

Example 2.7. (1) Let R be a noncommutative domain. Then R is Abelian. Let S be a semigroup with multiplication $st = 0$ for each $s, t \in S$ and $M = S^1$. Then R is not M -central Armendariz.

(2) Let $M = (\mathbb{N} \cup \{0\}, +)$ and

$$R = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} : a, b, c, d \in \mathbb{Z} \text{ and } a \equiv d, b \equiv c \equiv 0 \pmod{2} \right\}.$$

By [2, Example 2.2], R is an Abelian ring which is not M -central Armendariz.

Let (M, \leq) be an ordered monoid. If for any $g, g', h \in M$, $g < g'$ implies that $gh < g'h$ and $hg < hg'$, then (M, \leq) is called a strictly ordered monoid.

A ring R is called right principal projective (it or simply, right p.p.-ring) if the right annihilator of an element of R is generated by an idempotent. Clearly, M -Armendariz rings are M -central Armendariz. In the next theorem, we prove that the converse is true if the ring is a right p.p.-ring.

Theorem 2.8. *Let R be a right p.p.-ring and M be a strictly totally ordered monoid. If R is M -central Armendariz, then R is M -Armendariz.*

Proof. Let $\alpha = a_1g_1 + a_2g_2 + \dots + a_mg_m \in R[M]$ and $\beta = b_1h_1 + b_2h_2 + \dots + b_nh_n \in R[M]$ be such that $\alpha\beta = 0$, $g_1 < g_2 < \dots < g_m$ and $h_1 < h_2 < \dots < h_n$. We will use transitive induction on strictly totally ordered set (M, \leq) to show that $a_ib_j = 0$ for each $1 \leq i \leq m$ and $1 \leq j \leq n$. Since $h_1 \leq h_j$ for all $1 \leq j \leq n$, $g_1 < g_j$ implies $g_1h_1 < g_ih_1 \leq g_ih_j$ for each $1 \leq i \leq m$. Hence, if there exists $1 \leq i \leq m$ and $1 \leq j \leq n$ such that $g_ih_j = g_1h_1$, then $g_1 = g_i$ and $h_j = h_1$. Therefore $a_1b_1 = 0$.

Now, suppose that $w \in M$ is such that for any g_i and h_j with $g_ih_j < w$, $a_ib_j = 0$. We will show that $a_ib_j = 0$ for any g_i and h_j with $g_ih_j = w$. Set $X = \{(g_i, h_j) : g_ih_j = w\}$. Then X is a finite set. We write X as $\{(g_{i_t}, h_{j_t}) : t = 1, 2, \dots, k\}$ such that $g_{i_1} < g_{i_2} < \dots < g_{i_k}$. Since M is cancellative, $g_{i_1} = g_{i_2}$ and $g_{i_1}h_{j_1} = g_{i_2}h_{j_2} = w$ imply $h_{j_1} = h_{j_2}$. Since \leq is a strict order, $g_{i_1} < g_{i_2}$ and $g_{i_1}h_{j_1} = g_{i_2}h_{j_2} = w$ imply $h_{j_2} < h_{j_1}$.

Thus we have $h_{j_k} < \dots < h_{j_2} < h_{j_1}$. Now

$$\sum_{(g_i, h_j) \in X} a_i b_j = \sum_{t=1}^k a_{i_t} b_{j_t} = 0. \quad (2.1)$$

For any $t \geq 2$, $g_{i_1} h_{j_t} < g_{i_t} h_{j_t} = w$, and so by induction hypothesis, we have $a_{i_1} b_{j_t} = 0$. Since R is a right p.p.-ring, $r_R(a_{i_t}) = e_t R$ for some idempotent element e_t of R . Since R is M -central Armendariz, R is Abelian by Proposition 2.5. Hence $e_t \in C(R)$. Since $b_{j_t} \in r_R(a_{i_1})$ for each $t \geq 2$, $b_{j_t} e_1 = b_{j_t}$. By multiplying (2.1) by e_1 from the right, we have $a_{i_1} b_{j_1} e_1 + a_{i_2} b_{j_2} e_1 + \dots + a_{i_k} b_{j_k} e_1 = a_{i_1} e_1 b_{j_1} + a_{i_2} b_{j_2} + \dots + a_{i_k} b_{j_k} = 0$. Hence

$$a_{i_2} b_{j_2} + \dots + a_{i_k} b_{j_k} = 0. \quad (2.2)$$

For any $t \geq 3$, $g_{i_2} h_{j_t} < g_{i_t} h_{j_t} = w$. So by induction hypothesis, we have $a_{i_2} b_{j_t} = 0$. Hence $b_{j_t} e_2 = b_{j_t}$. By multiplying (2.2) by e_2 from the right, we have $a_{i_3} b_{j_3} + \dots + a_{i_k} b_{j_k} = 0$. Continuing this process yield $a_{i_k} b_{j_k} = 0$. Thus (2.2) has the form $a_{i_1} b_{i_1} + \dots + a_{i_{k-1}} b_{j_{k-1}} = 0$. As above, we have $a_{i_{k-1}} b_{j_{k-1}} = \dots = a_{i_2} b_{j_2} = a_{i_1} b_{j_2} = 0$.

Therefore by transitive induction, $a_i b_j = 0$ for any i, j . Thus R is M -central Armendariz. \square

In the following example, it is shown that the condition "right p.p.-ring" in Theorem 2.8 is not superfluous.

Example 2.9. Let \mathbb{Z}_2 be the field of integers modulo 2 and

$$R = \{a_0 + a_1 i + a_2 j + a_3 k : a_i \in \mathbb{Z}_2 \text{ for } i = 0, 1, 2, 3\}$$

be the Hamiltonian quaternions over \mathbb{Z}_2 . Then R is not a p.p.-ring by [13, Example 1]. Since R is a commutative ring, it is M -central Armendariz for each monoid M . Let M be a monoid with $|M| \geq 2$, $e \neq g \in M$ and $\alpha = (1+i)e + (1+j)g$. Then $\alpha^2 = 0$, but $(1+i)(1+j) \neq 0$ which implies that R is not M -Armendariz.

It was shown in [2, Theorem 2.6] that if I is a reduced ideal of R such that R/I is a central Armendariz ring, then R is central Armendariz. Here we have the following result, which is a generalization of this Theorem.

Theorem 2.10. *Let M be a strictly totally ordered monoid and I is an ideal of R . If I is a reduced ring and R/I is M -central Armendariz, then R is M -central Armendariz.*

Proof. Let $a, b \in R$ and $ab = 0$. Then by a similar argument in the proof of [2, Theorem 2.6], we have $bIa = aIb = 0$. Let $\alpha = a_1 g_1 + a_2 g_2 + \dots + a_m g_m \in R[M]$ and $\beta = b_1 h_1 + b_2 h_2 + \dots + b_n h_n \in R[M]$ be

such that $\alpha\beta = 0$, $g_1 < g_2 < \dots < g_m$ and $h_1 < h_2 < \dots < h_n$. We will use transitive induction on strictly totally ordered set (M, \leq) to show that $a_i I b_j = b_j I a_i = 0$ for each $1 \leq i \leq m$ and $1 \leq j \leq n$. By analogy with the proof of Theorem 2.8, we can show $a_1 b_1 = 0$. Hence $a_1 I b_1 = b_1 I a_1 = 0$.

Now, suppose that $w \in M$ is such that for any g_i and h_j with $g_i h_j < w$, $a_i I b_j = b_j I a_i = 0$. We will show that $a_i I b_j = b_j I a_i = 0$ for any g_i and h_j with $g_i h_j = w$. Set $X = \{(g_i, h_j) : g_i h_j = w\}$. Then X is a finite set. We write X as $\{(g_{i_t}, h_{j_t}) : t = 1, 2, \dots, k\}$ such that $g_{i_1} < g_{i_2} < \dots < g_{i_k}$. Since M is cancellative, $g_{i_1} = g_{i_2}$ and $g_{i_1} h_{j_1} = g_{i_2} h_{j_2} = w$ imply $h_{j_1} = h_{j_2}$. Since \leq is a strict order, $g_{i_1} < g_{i_2}$ and $g_{i_1} h_{j_1} = g_{i_2} h_{j_2} = w$ imply $h_{j_2} < h_{j_1}$. Thus we have $h_{j_k} < \dots < h_{j_2} < h_{j_1}$. Now

$$\sum_{(g_i, h_j) \in X} a_i b_j = \sum_{t=1}^k a_{i_t} b_{j_t} = 0. \quad (2.3)$$

For any $t \geq 2$, $g_{i_1} h_{j_t} < g_{i_t} h_{j_t} = w$, and so by induction hypothesis, we have $a_{i_1} I b_{j_t} = b_{j_t} I a_{i_1} = 0$. Since $a_{i_1} I a_{i_t} b_{j_t} \subseteq a_{i_1} I b_{j_t} = 0$, $a_{i_1} I a_{i_t} b_{j_t} = 0$ for $t \geq 2$. By multiplying (2.3) from the left by $a_{i_1} I$, we have $a_{i_1} I a_{i_1} b_{j_1} = 0$, because $a_{i_1} I a_{i_t} b_{j_t} = 0$ for $t \geq 2$. Therefore $(b_{j_1} I a_{i_1})^3 = 0$ because $a_{i_1} b_{j_1} I a_{i_1} b_{j_1} \subseteq a_{i_1} I a_{i_1} b_{j_1} = 0$. Since I is reduced, $b_{j_1} I a_{i_1} = 0$. Also $a_{i_1} I b_{j_1} = 0$.

For any $t \geq 3$, $g_{i_2} h_{j_t} < g_{i_t} h_{j_t} = w$. So by induction hypothesis, we have $a_{i_2} I b_{j_t} = b_{j_t} I a_{i_2} = 0$. Since $a_{i_1} I b_{j_1} = 0$, $(a_{i_2} I a_{i_1} b_{j_1})^2 = 0$. Hence $a_{i_2} I a_{i_1} b_{j_1} = 0$. By multiplying (2.3) from left to $a_{i_2} I$, we have $a_{i_2} I a_{i_2} b_{j_2} = 0$. As above $a_{i_2} I b_{j_2} = b_{j_2} I a_{i_2} = 0$. By continuing this process, we have $a_{i_t} I b_{j_t} = b_{j_t} I a_{i_t} = 0$ for each $t = 1, 2, \dots, k$. Therefore by transfinite induction, $a_i I b_j = b_j I a_i = 0$.

Note that in $(R/I)[M]$, $(\overline{a_1} g_1 + \dots + \overline{a_n} g_m)(\overline{b_1} h_1 + \dots + \overline{b_m} h_m) = 0$. Since R/I is M -central Armendariz, $\overline{a_i} \overline{b_j} \in C(R/I)$. Thus $a_i b_j r - r a_i b_j \in I$ for any i, j and $r \in R$. Therefore $(a_i b_j r - r a_i b_j)^3 = 0$. As I is reduced, $a_i b_j r = r a_i b_j$ for each $r \in R$ and i, j . Hence R is M -central Armendariz. \square

Recall that a monoid M is called torsion-free if the following property holds: if $g, h \in M$ and $k \geq 1$ are such that $g^k = h^k$, then $g = h$.

Corollary 2.11. *Let M be a commutative, cancellative and torsion-free monoid. If R/I is M -central Armendariz for some ideal I of R and I is a reduced ring, then R is M -central Armendariz.*

Proof. If M is commutative, cancellative and torsion-free, then by [21] there exists a compatible strict total ordered \leq on M . Now the results follows from Theorem 2.10. \square

Remark 2.12. Let R be any ring and $n \geq 2$. Consider the ring $M_n(R)$ of $n \times n$ matrices and the ring $T_n(R)$ of $n \times n$ upper triangular matrices over R . Then the rings $M_n(R)$ and $T_n(R)$ are not abelian. By Proposition 2.5, these rings are not M -central Armendariz for each monoid M .

The next example shows that if I is an ideal of R , R/I and I are M -central Armendariz for a u.p.- monoid M , then R is not M -central Armendariz in general.

Example 2.13. Let F be a field and consider $R = T_2(F)$, which is not M -central Armendariz, for a u.p.- monoid M by Remark 2.12 with $|M| \geq 2$. It can be seen that R/I and I are M -central Armendariz for some nonzero proper ideal I of R . Assume that $I = \begin{pmatrix} F & F \\ 0 & 0 \end{pmatrix}$. Then $R/I \cong F$ and so R/I is M -Armendariz. Hence R/I is M -central Armendariz. We prove that I is M -central Armendariz. Now let $\alpha = \sum_{i=1}^n \begin{pmatrix} a_i & b_i \\ 0 & 0 \end{pmatrix} g_i$ and $\beta = \sum_{j=1}^m \begin{pmatrix} c_j & d_j \\ 0 & 0 \end{pmatrix} h_j$ be nonzero elements of $I[M]$ such that $\alpha\beta = 0$. From the isomorphism $T_2(F)[M] \cong T_2(F[M])$ defined by:

$$\sum_{i=1}^n \begin{pmatrix} a_i & b_i \\ 0 & c_i \end{pmatrix} g_i \longrightarrow \begin{pmatrix} \sum_{i=1}^n a_i g_i & \sum_{i=1}^n b_i g_i \\ 0 & \sum_{i=1}^n c_i g_i \end{pmatrix}$$

we have $\alpha_1\beta_1 = \alpha_1\beta_2 = 0$, where $\alpha_1 = \sum_{i=1}^n a_i g_i$, $\beta_1 = \sum_{j=1}^m c_j h_j$ and $\beta_2 = \sum_{j=1}^m d_j h_j \in F[M]$. As F is reduced and M is a u.p.-monoid, F is M -Armendariz by [19, Proposition 1.1]. Hence $a_i c_j = a_i d_j = 0$ for each i, j . Therefore

$$\begin{pmatrix} a_i & b_i \\ 0 & 0 \end{pmatrix} \begin{pmatrix} c_j & d_j \\ 0 & 0 \end{pmatrix} = 0$$

for each i, j . This implies that I is M -Armendariz, and so it is M -central Armendariz.

Lemma 2.14. *Let M be a cyclic group of order $n \geq 2$ and R a non-commutative ring with $0 \neq 1$. Then R is not M -central Armendariz.*

Proof. Assume that $M = \{e, g, g^2, \dots, g^{n-1}\}$ and $a \in R - C(R)$. Let $\alpha = ae + ag + ag^2 + \dots + ag^{n-1}$ and $\beta = 1e + (-1)g$. Then $\alpha\beta = 0$. Since $a \notin C(R)$, R is not M -central Armendariz. \square

The proof of the next lemma is straightforward.

Lemma 2.15. *Let M be a monoid and N a submonoid of M . If R is M -central Armendariz, then R is N -central Armendariz.*

Proposition 2.16. *Let M be a cancellative monoid and N an ideal of M . If R is N -central Armendariz, then R is M -central Armendariz.*

Proof. Let $\alpha = a_1g_1 + a_2g_2 + \dots + a_mg_m \in R[M]$ and $\beta = b_1h_1 + b_2h_2 + \dots + b_nh_n \in R[M]$ be such that $\alpha\beta = 0$. Let $g \in N$. Then $g_i g, h_j g \in N$ for each i, j . Also $g_s g \neq g_t g$ and $h_s g \neq h_t g$ for each $s \neq t$. Now from $(\alpha)g(\beta)g = (\sum_{i=1}^m a_i g_i g)(\sum_{j=1}^n b_j h_j g) = 0$, we have $a_i b_j \in C(R)$, because R is N -central Armendariz. \square

Let $T(G)$ be the set of elements of finite order in an Abelian group G . Then $T(G)$ is a fully invariant subgroup of G . G is said to be torsion-free if $T(G) = \{e\}$.

Proposition 2.17. *Suppose G is a finitely generated Abelian group. Then G is torsion-free if and only if there exists a ring R with $|R| \geq 2$ such that R is G -central Armendariz.*

Proof. See [19, Theorem 1.14]. \square

In [15], Baer rings are introduced as rings in which the right (left) annihilator of every nonempty subset is generated by an idempotent. We end this paper with some applications of M -central Armendariz rings to show there is a strong connection between Baer and p.p.-rings with their monoid rings.

Theorem 2.18. *Let M be a strictly totally ordered monoid with $|M| \geq 2$ and R an M -central Armendariz ring. Then R is right p.p.-ring if and only if $R[M]$ is right p.p.-ring.*

Proof. Let R be a right p.p.-ring. By Theorem 2.8, R is M -Armendariz, because R is M -central Armendariz. Hence $R[M]$ is a right p.p.-ring by [19, Theorem 3.4].

Conversely, let $R[M]$ be a right p.p.-ring and $a \in R$. Then there exists an idempotent $f = f_1g_1 + f_2g_2 + \dots + f_mg_m \in R[M]$ such that $r_{R[M]}(a) = fR[M]$. We can suppose that $g_1 < g_2 < \dots < g_m$. If there exist $1 \leq i, j \leq m$ such that $g_i g_j = g_1^2$, then $g_1 \leq g_i$ and $g_1 \leq g_j$. If $g_1 < g_i$, then $g_1^2 < g_i g_1 \leq g_i g_j = g_1^2$, a contradiction. Thus $g_1 = g_i$. Similarly, $g_1 = g_j$. Hence from $(1 - f)f = 0$, we have $(1 - f_1)f_1 = 0$. Therefore $f_1^2 = f_1$. As $afR[M] = 0$, $af_1g_1 + \dots + af_mg_m = 0$. Since $g_i \neq g_j$ for each $1 \leq i \neq j \leq m$, $af_1 = 0$. Thus $f_1R \subseteq r_R(a)$. Now, let $r \in r_R(a)$. Then $r \in fR[M]$. This implies that $(1 - f)r = 0$ and so $(1 - f_1)r = 0$. Hence $r \in f_1R$. Therefore $r_R(a) = f_1R$. Thus R is a right p.p.-ring. \square

Theorem 2.19. *Let M be a strictly totally ordered monoid with $|M| \geq 2$ and R an M -central Armendariz ring. Then R is a Baer ring if and only if $R[M]$ is a Baer ring.*

Proof. Let R be a Baer ring. By Theorem 2.8, R is M -Armendariz, because R is M -central Armendariz. Hence $R[M]$ is a Baer ring by [19, Theorem 3.5].

Conversely, let $R[M]$ be a Baer ring and W be a subset of R . Since $R[M]$ is Baer, there exists $e^2 = e = e_1g_1 + e_2g_2 + \dots + e_mg_m \in R[M]$ such that $r_{R[M]}(W) = eR[M]$. Similar the proof of Theorem 2.18, $e_1^2 = e_1$. As $we = 0$ for each $w \in W$, $we_1 = 0$ for each $w \in W$. Therefore $e_1R \subseteq r_R(W)$. Now, let $r \in r_R(W)$. Then from $r_R(W) \subseteq r_{R[M]}(W) = eR[M]$ we have $(1 - e)r = 0$. Hence $r \in e_1R$. Thus $r_R(W) = e_1R$ and so R is Baer. \square

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